Oxford Cambridge and RSA

A Level Mathematics A

H240/01 Pure Mathematics

Question Set 1

1

The points A and B have coordinates (1, 5) and (4, 17) respectively. Find the equation of the straight line which passes through the point (2, 8) and is perpendicular to AB. Give your answer in the form $ax + by = c$, where a, b and c are constants. **[4]**

1. $A\ (1, 5)$
$B\ (4, 17)$

gradient AB: $\dfrac{17-5}{4-1} = 4$

perpendicular gradient $= -\frac{1}{4}$

as $4 \times -\frac{1}{4} = -1$

$(2, 8)$
gradient $= -\frac{1}{4}$

$$y - y_1 = m(x - x_1)$$
$$\left[\, y - 8 = -\tfrac{1}{4}(x-2) \,\right] \times 4$$
$$4y - 32 = -(x-2)$$
$$4y - 32 = -x + 2$$
$$x + 4y = 34$$

(a) Use the trapezium rule, with four strips each of width 0.5, to estimate the value of

$$\int_0^2 e^{x^2}\,dx$$

giving your answer correct to 3 significant figures. **[3]**

2. a) $x = 0.0 \sim 0.5 \Rightarrow 0.5$ $\quad y = e^0, e^{0.25}$
$x = 0.5 \sim 1.0 \Rightarrow 0.5$ $\quad y = e^{0.25}, e^1$
$x = 1.0 \sim 1.5 \Rightarrow 0.5$ $\quad y = e^1, e^{2.25}$
$x = 1.5 \sim 2.0 \Rightarrow 0.5$ $\quad y = e^{2.25}, e^4$

$$\int_0^2 e^{x^2}\,dx = \frac{1}{2} \times 0.5 \times \left(e^0 + e^4 + 2(e^{0.25} + e^1 + e^{2.25})\right)$$
$$= \frac{1}{2} \times 0.5 \times 82.578 = 20.6$$

(b) Explain how the trapezium rule could be used to obtain a more accurate estimate. **[1]**

b) - use more trapezia, of a narrower width, over the same interval

3 The cubic polynomial $f(x)$ is defined by $f(x) = 2x^3 - 7x^2 + 2x + 3$.

(a) Given that $(x-3)$ is a factor of $f(x)$, express $f(x)$ in a fully factorised form. **[3]**

3. $f(x) = 2x^3 - 7x^2 + 2x + 3$

a) $(x-3)$ so $f(3)$

$f(3) = 2(3)^3 - 7(3)^2 + 2(3) + 3 = 0$

since $f(3) = 0$, $(x-3)$ is a factor of $f(x)$

(b) Sketch the graph of $y = f(x)$, indicating the coordinates of any points of intersection with the axes. **[2]**

b)

$$\begin{array}{r|l} & 2x^2 - x - 1 \\ \hline x-3 & 2x^3 - 7x^2 + 2x + 3 \\ & \underline{2x^3 - 6x^2} \\ & \quad -x^2 + 2x + 3 \\ & \quad \underline{-x^2 + 3x} \\ & \qquad -x + 3 \\ & \qquad \underline{-x + 3} \\ & \qquad\quad 0 \end{array}$$

$\Rightarrow$ factorise $\rightarrow (2x+1)(x-1)$

$f(x) = (2x+1)(x-3)(x-1)$

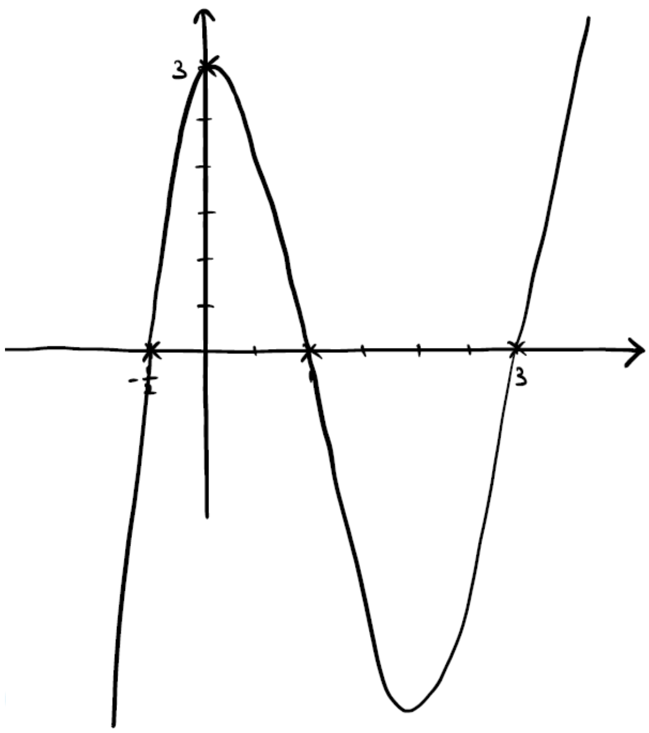
when $f(x) = 0$ / $y = 0$

$x = -\frac{1}{2}$ or 3 or 1

when $x = 0$

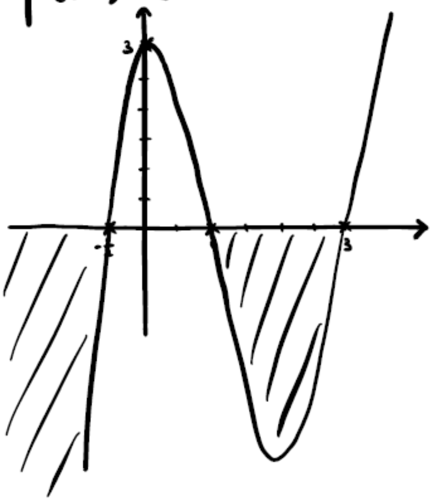
$y = 3$

$(-\frac{1}{2}, 0)$
$(3, 0)$
$(1, 0)$
$(0, 3)$



(c) Solve the inequality $f(x) < 0$, giving your answer in set notation. [2]

c) $f(x) < 0$



$x < -\frac{1}{2}$, $1 < x < 3$

$\{x : x < -\frac{1}{2}\} \cup \{x : 1 < x < 3\}$

for < 0

(d) The graph of $y = f(x)$ is transformed by a stretch parallel to the x-axis, scale factor $\frac{1}{2}$.
Find the equation of the transformed graph. [2]

d) $y = f(x) = (2x+1)(x-3)(x-1)$

$\rightarrow \leftarrow$ curve by S.F of $\frac{1}{2}$ so $x \uparrow$ by $\times 2$

$y = (4x+1)(2x-3)(2x-1)$

4 Chris runs half marathons, and is following a training programme to improve his times. His time for his first half marathon is 150 minutes. His time for his second half marathon is 147 minutes. Chris believes that his times can be modelled by a geometric progression.

(a) Chris sets himself a target of completing a half marathon in less than 120 minutes. Show that this model predicts that Chris will achieve his target on his thirteenth half marathon. **[4]**

4. $u_1 = 150$ $u_2 = 147$

a) $u_n = ar^{n-1}$

$147 = 150r^{2-1}$

$r = \frac{49}{50}$

$u_n < 120$

$150 \times \left(\frac{49}{50}\right)^{n-1} < 120$

$\ln\left[\left(\frac{49}{50}\right)^{n-1} < \frac{4}{5}\right]$

$(n-1)\ln\left(\frac{49}{50}\right) < \ln\left(\frac{4}{5}\right)$

change in sign as $\ln(\frac{49}{50})$ is negative

$n-1 > \frac{\ln(\frac{4}{5})}{\ln(\frac{49}{50})}$

$n-1 > 11.05$

$n > 12.05$

so first whole number after is 13

(b) After twelve months Chris has spent a total of 2974 minutes, to the nearest minute, running half marathons. Use this model to find how many half marathons he has run. **[3]**

b) $S_n = \frac{a(1-r^n)}{1-r}$

① $2974 = \frac{150\left(1-\left(\frac{49}{50}\right)^n\right)}{1-\left(\frac{49}{50}\right)}$

② $2974 = \frac{150\left(1-\left(\frac{49}{50}\right)^n\right)}{\frac{1}{50}}$

③ $\frac{1487}{25} = 150\left(1-\left(\frac{49}{50}\right)^n\right)$

④ $\frac{1487}{3750} = 1-\left(\frac{49}{50}\right)^n$

⑤ $\left(\frac{49}{50}\right)^n = \frac{2263}{3750}$

⑥ $n\ln\left(\frac{49}{50}\right) = \ln\left(\frac{2263}{3750}\right)$

⑦ $n = \frac{\ln\left(\frac{2263}{3750}\right)}{\ln\left(\frac{49}{50}\right)} = 25.0$

25 half marathons

[2]

(c) Give two reasons why this model may not be appropriate when predicting the time for a half marathon. [2]

c) - doesn't take into account possible variations in conditions
- assumes that times will continue to improve

5 In a science experiment a substance is decaying exponentially. Its mass, M grams, at time t minutes is given by $M = 300e^{-0.05t}$.

(a) Find the time taken for the mass to decrease to half of its original value. [3]

5. $M = 300e^{-0.05t}$

a) $M = 150$ for half

1st substance

$150 = 300e^{-0.05t}$

$\frac{150}{300} = e^{-0.05t}$

$\ln\frac{1}{2} = -0.05t \times \ln e$ (1)

$t = \frac{\ln\frac{1}{2}}{-0.05} = 13.9$ minutes

(b) Find the time at which both substances are decaying at the same rate. [8]

b) $m = m_0 e^{kt}$

$320 = 400e^{k \times 10}$

$e^{10k} = \frac{320}{400}$

$10k \times \ln e = \ln\frac{320}{400}$ (1)

$k = \frac{\ln\frac{320}{400}}{10} = -0.0223$

$m = 400e^{-0.0223t}$

1st substance: $\frac{dM}{dt} = -15e^{-0.05t}$

2nd substance: $\frac{dm}{dt} = -8.93e^{-0.0223t}$

$-15e^{-0.05t} = -8.93e^{-0.0223t}$

$\frac{-15}{-8.93} = \frac{e^{-0.0223t}}{e^{-0.05t}}$

$e^{(-0.0223t)-(-0.05t)} = 1.681$

$e^{0.0277t} = 1.681$

$0.0277t \times \ln e = \ln 1.681$

$0.0277t = \ln 1.681$

$t = \frac{\ln 1.681}{0.0277} = 18.75$ minutes

6 A scientist is attempting to model the number of insects, N, present in a colony at time t weeks. When $t = 0$ there are 400 insects and when $t = 1$ there are 440 insects.

(a) A scientist assumes that the rate of increase of the number of insects is inversely proportional to the number of insects present at time t.

(i) Write down a differential equation to model this situation. **[1]**

6. $t = 0$ $N = 400$

$t = 1$ $N = 440$

a) rate of increase of number of insects $\propto \dfrac{1}{\text{number of insects present}}$

(i) $\dfrac{dN}{dt} = \dfrac{k}{N}$

(ii) Solve this differential equation to find N in terms of t. **[4]**

(ii) $\dfrac{dN}{dt} = \dfrac{k}{N}$

$N \, dN = k \, dt$

$\int N \, dN = \int k \, dt$ constant so can be moved out

$\int N \, dN = k \int 1 \, dt$

$\frac{1}{2}N^2 + c = kt$

when $t = 0$ $\frac{1}{2}(400)^2 + c = k(0)$

$80000 + c = 0$

$c = -80000$

$\frac{1}{2}N^2 - 80000 = kt$

when $t = 1$ $\frac{1}{2}(440)^2 - 80000 = k(1)$

$k = 16800$

$\frac{1}{2}N^2 - 80000 = 16800t$

$\frac{1}{2}N^2 = 16800t + 80000$

$N^2 = 33600t + 160000$

$N = \sqrt{33600t + 160000}$ positive as number of insects

6

(b) In a revised model it is assumed that $\frac{dN}{dt} = \frac{N^2}{3988e^{0.2t}}$. Solve this differential equation to find N in terms of t. **[6]**

b) $\frac{dN}{dt} = \frac{N^2}{3988e^{0.2t}}$

$\frac{1}{N^2}\,dN = \frac{1}{3988e^{0.2t}}\,dt$

$\int N^{-2}\,dN = \int \frac{1}{3988} e^{-0.2t}\,dt$

$\int N^{-2}\,dN = \frac{1}{3988} \int e^{-0.2t}\,dt$

$-N^{-1} + c = \frac{1}{3988} \times -5e^{-0.2t}$ constant so not affected

$\frac{-3988}{N} + c = -5e^{-0.2t}$

when $t = 0$ $\quad \frac{-3988}{400} + c = -5e^{-0.2\times 0} = -5e^{0} = -5$

$c = 4.97$

$\left[-\frac{3988}{N} + 4.97 = -5e^{-0.2t}\right] \times N$

$-3988 + 4.97N = -5Ne^{-0.2t}$

$4.97N + 5Ne^{-0.2t} = 3988$

$(4.97 + 5e^{-0.2t})N = 3988$

$N = \frac{3988}{4.97 + 5e^{-0.2t}}$

(c) Compare the long-term behaviour of the two models. **[2]**

c) - 1st model predicts that population will continue to increase
- 2nd model predicts that population will tend towards a limit of 802

Total Marks for Question Set 1: 50 Marks